ONE CASE OF THE EQUILIBRIUM OF A SYSTEM OF RADIAL CRACKS

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The plane-elasticity problem of the equilibrium of a system of uniformly distributed cracks of equal length intersecting in a single point is considered. The system is located with concentrated forces applied to the tips of the wedges cut out by the cracks and acting along the bisectrices of the wedges. An analytical expression is found for the singularity coefficient of the stress field at the tip of the cracks.

1. The plane-elasticity theory of the equilibrium of a system of uniformly distributed radial cracks of equal length emerging into a round cavity was considered in [1]. This problem is of interest from the point of view of the rupture of brittle solids by the explosion of long cylindrical charges. There may be two kinds of loading – the first in which a constant pressure is applied both to the boundary of the cavity and to the borders of the cracks, and the second in which a constant pressure is applied to the boundary of the cavity while the borders of the cracks are stress-free. A numerical solution was given in [1] for the first mode of loading in the case of one or two cracks.

The limiting case of the problem for an infinitely small radius of the cavity was considered in [2], in which an analytical solution was given for the equilibrium of a system of uniformly distributed cracks of equal length intersecting at a single point and loaded with a constant pressure applied to the borders of the cracks. An approximate analytical solution of this problem was obtained in [3].

In this paper we shall consider the equilibrium of a similar system of cracks loaded with concentrated forces applied to the tips of the wedges cut out by the cracks and directed along the bisectrices of these wedges. This presentation of the problem corresponds to that of [1] for the second mode of loading in which the crack length is much greater than the radius of the cavity.

Using the Wiener–Hopf method we shall obtain a solution for the paired integral equations derived in [2], but with different boundary conditions. We shall give an analytical solution for the singularity coefficient of the stress field at the tip of the crack.

Let there be n radial cracks of unit length in a thin infinite elastic plate. The angles between neighboring cracks are $2\pi/n$. The borders of the cracks are stress-free. To the tips of the wedges formed by the cracks we apply concentrated forces Q, acting along the bisectrices to the angles of these wedges.

The symmetry of the problem allows us to consider one such wedge on its own. We introduce a polar coordinate system (r, θ). Let the wedge occupy a region $0 < r < \infty$, $|\theta| \leq \pi/n$.

It follows from the symmetry of the problem that

$$\begin{aligned} &\sigma_{r\theta} = 0 \quad \text{for} \quad |\theta| = \pi/n, \quad 0 < r < \infty \\ &v = 0 \quad \text{for} \quad |\theta| = \pi/r, \quad 1 < r < \infty \end{aligned} \tag{1.1}$$

Here $\sigma_{r\theta}$ is the stress-tensor component, and v is the tangential component of the displacement vector.

In order to describe the action of the concentrated force applied to the tip of the wedge, let us consider the action of such a force on a free infinite wedge, on the boundaries of which the conditions $\sigma_{\theta\theta} = 0$,

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$\sigma_{r\theta}=0$ are satisfied for $|\theta|=\pi/n$. The force Q acts along the $\theta=0$ axis. The solution of this problem [4] is described by an Airy function:

$$\Psi = -[Q\theta\, r \sin\theta][2\pi/n + \sin 2\pi/n]^{-1}. \tag{1.2}$$

The values of the tangential displacement at $\theta=\pi/n$ are expressed by the equations

$$\begin{gathered} Ev = A \ln r - B \\ A = 2Q(\sin\pi/n)\,[2\pi/n - \sin 2\pi/n]^{-1} \end{gathered} \tag{1.3}$$

$$B = Q[2\pi/n + \sin 2\pi/n]\left[(1-\nu)\sin 2\pi/n - (1-\nu)\frac{\pi}{n}\cos 2\pi/n\right] - U_0 \sin\pi/n.$$

Here U_0 is an arbitrary constant, E is Young's modulus, ν is the Poisson coefficient. We subtract (1.2) from the equations describing the problem under consideration. The boundary conditions of the difference problem may be derived from Eqs. (1.1) and (1.3):

$$\begin{gathered} \sigma_{r\theta} = 0 \quad \text{for} \quad |\theta| = \pi/n, \quad 0 < r < \infty \\ \sigma_{\theta\theta} = 0 \quad \text{for} \quad |\theta| = \pi/n, \quad 0 < r < 1 \\ v = \pm(A \ln r - B) \quad \text{for} \quad \theta = \pm\pi/n, \quad 1 < r < \infty \\ Q = 2\sin\pi/n \int_1^\infty \sigma_{\theta\theta}\, dr. \end{gathered} \tag{1.4}$$

2. Let us consider the solution of the problem (1.4). The following expression was obtained earlier [2] for the transform Ψ after applying a Mellin transformation to the biharmonic equation for the Airy function Ψ, allowing for the symmetry of the problem and the condition (1.1):

$$\overline{\Psi}(\theta, s) = \int_0^\infty \Psi(r, \theta)\, r^{s-2}\, dr = F(\theta, s)\,\Psi(s) \tag{2.1}$$

$$F(\theta, s) = \frac{1}{4}\left\{\frac{(s+1)\cos[(s-1)\theta]}{(s-1)\sin[(s-1)\pi/n]} - \frac{\cos[(s+1)\theta]}{\sin[(s+1)\pi/n]}\right\}.$$

We then obtain the following expressions for $\sigma_{\theta\theta}$ and $\bar{u}$ at the boundaries of the wedge when $\theta=\pi/n$

$$\bar{\sigma}_{\theta\theta}(s) = \int_0^\infty \sigma_{\theta\theta}(r)\, r^s\, dr = k(s)\,\psi(s) \tag{2.2}$$

$$E\bar{v}(s) = E\int_0^\infty v(r)\, r^{s-1}\, dr = \psi(s)$$

$$k(s) = \frac{s}{4}\left\{\frac{\sin(2s\pi/n) + s\sin 2\pi/n}{\sin[(s-1)\pi/n]\sin[(s+1)\pi/n]}\right\}.$$

Let us introduce the functions

$$\begin{gathered} \sigma_+(s) = \int_0^1 \sigma_{\theta\theta}(r)\, r^s\, dr, \quad \sigma_-(s) = \int_1^\infty \sigma_{\theta\theta}(r)\, r^s\, dr \\ v_+(s) = \int_0^1 v(r)\, r^{s-1}\, dr, \quad v_-(s) = \int_1^\infty v(r)\, r^{s-1}\, dr. \end{gathered} \tag{2.3}$$

It follows from conditions (2.2) and (2.3) that

$$\sigma_-(s) = 0, \quad v_-(s) = A/s^2 - B/s. \tag{2.4}$$

If we assume that $v(r) \sim r^\varepsilon$ as $r \to 0$, where $\varepsilon > 0$ [the constant part of v(r) may be eliminated by an appropriate choice of U_0] and $\sigma_{\theta\theta} \sim r^{-1}$ as $r \to \infty$, then in relation to the functions σ_- and v_+ we may conclude [5] that $\sigma_-(s)$ is a regular function of the complex variable s for Re s < 0, while $v_+(s)$ is regular for Re $s > -\varepsilon$. Eliminating $\psi(s)$ from (2.2), we obtain the following functional equation for $v_+(s)$, $\sigma_-(s)$

$$\sigma_- = -k(s)E(v_+ + v_-). \tag{2.5}$$

If we factorize the function k(s) as in [2], we may express k(s) in the form

$$\begin{gathered} k(s) = K_+K_- \\ K_+ = \frac{s^2F_+}{1+s}, \quad K_- = \frac{-c(n)F_-}{1-s} \\ c(n) = \frac{2\pi/n + \sin 2\pi/n}{4\sin^2\pi/n} \\ F_+(s) = F_-(s) = \prod_{m=1}^\infty \frac{(1 - s/a_m)(1 - s/\bar{a}_m)}{[1 + s/(mn+1)][1 - s/(mn-1)]}. \end{gathered} \tag{2.6}$$

Here a_m are the roots of the equation $\sin(2s\pi/n)+s[\sin(2\pi/n)]=0$ in the first square of the plane s. The function K_+ is regular for Re $s>-1$, K_- is regular for Re $s<1$ and has no zeros in this region. Let us divide Eq. (2.5) by K_-:

$$\sigma_- / K_- = -K_+ E v_+ - K_+ E \iota_- . \tag{2.7}$$

The left-hand side of this equation is regular for Re $s<0$, the right-hand side is regular for Re $s>-\varepsilon$.

Let us see how these functions behave at infinity:

$$\sigma_- \sim |s|^{-1/2}, \quad K_- \sim |s|^{-1/2} \quad \text{as} \quad s \to -\infty$$
$$v_+ \sim s^{-3/2}, \quad K_+ \sim s^{3/2} \quad \text{as} \quad s \to +\infty .$$

(for σ_- and v_+ the order of magnitude at infinity is determined by the known behavior of the solution at the tip of the crack for $r=1$.) It follows from Liouville's theorem and Eq. (2.4) that

$$\sigma_- / K_- = \sigma_-(0) / K_-(0) = Q / 2c(n) \sin \pi / n . \tag{2.8}$$

Equations (2.8), (2.6), and (2.2) determine the form of the functions σ_- and $\psi(s)$; we may express all the parameters of the stressed state in the form of contour integrals of the reciprocal Mellin transformation in terms of these.

Let us find the singularity coefficient for the stresses at the tip of the crack. According to [5], we find that if $\sigma_{\theta\theta} \sim N(r-1)^{1/2}$ as $r \to 1+0$, then $\sigma_- \sim N\Gamma(1/2)\ |s|^{1/2}$ as $s \to -\infty$. Thus, in order to find N it is sufficient to determine the behavior of σ_- as $s \to -\infty$. If we use the equation [2]

$$F_-(s) = \frac{\Gamma^2(3/4)\,\Gamma(1+1/n-s/n)\,\Gamma(1-1/n-s/n)}{\Gamma^2(3/4-s/n)\,\Gamma(1+1/n)\,\Gamma(1-1/n)} D(s)$$

$$D(s) = \prod_{m=1}^{\infty} \frac{(1-s/a_m)(1-s/\bar{a}_m)}{[1-s/(mn-n/4)]}$$

we find that as $s \to -\infty$

$$\sigma_- \approx \frac{Q}{2\sin\pi/n} \frac{\Gamma^2(3/4)\,D_0(n)}{\Gamma(1+1/n)\,\Gamma(1-1/n)} \sqrt{\frac{1}{-sn}}$$

$$D_0(n) = \lim_{s\to\infty} D(s) = \sqrt{\frac{n}{2c(n)}} \frac{\Gamma(1+1/n)\,\Gamma(1-1/n)}{\Gamma^2(3/4)} .$$

For a system of n cracks of length l we obtain

$$N = Q(2\pi)^{-1/2}\ [(2\pi/n + \sin 2\pi/n)l]^{-1/2} . \tag{2.9}$$

For $n=2$ this expression coincides with the coefficient of the stress-field singularity in the case of a plane crack of length $2l$ stretched by concentrated forces Q applied to the middle points of its borders and acting along the normal to the crack [6]. Using Eq. (2.9) we obtain the following expression for the stress-field singularity coefficient at the tip of the crack in the case of a system of n radial free cracks of length l emerging into a cavity of radius r_0 loaded with a pressure p:

$$N = pr_0\sqrt{2\pi}n^{-1}[(2\pi/n + \sin 2\pi/n)l]^{-1/2}, \qquad l \gg r_0 . \tag{2.10}$$

LITERATURE CITED

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